

Solving simultaneously Thue equations in the almost totally imaginary case

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Abstract

Let α be an algebraic number of degree $d \geq 3$ having at most one real conjugate and let K be the algebraic number field $\mathbf{Q}(\alpha)$. For any unit ε of K such that $\mathbf{Q}(\alpha\varepsilon) = K$, we consider the irreducible polynomial $f_\varepsilon(X) \in \mathbf{Z}[X]$ such that $f_\varepsilon(\alpha\varepsilon) = 0$. Let $F_\varepsilon(X, Y) = Y^d f_\varepsilon(X/Y) \in \mathbf{Z}[X, Y]$ be the associated binary form. For each positive integer m , we exhibit an effectively computable bound for the solutions (x, y, ε) of the diophantine equation $|F_\varepsilon(x, y)| \leq m$.

1 The main theorem

Let α be an algebraic number of degree $d \geq 3$ over \mathbf{Q} . Denote by K the algebraic number field $\mathbf{Q}(\alpha)$, by $f \in \mathbf{Z}[X]$ the irreducible polynomial of α over \mathbf{Z} , by \mathbf{Z}_K^\times the unit group of K . For any unit $\varepsilon \in \mathbf{Z}_K^\times$ such that $\mathbf{Q}(\alpha\varepsilon) = K$, we denote by $f_\varepsilon(X) \in \mathbf{Z}[X]$ the irreducible polynomial of $\alpha\varepsilon$ over \mathbf{Z} (unique if we require the leading coefficient to be > 0) and by $F_\varepsilon(X, Y) \in \mathbf{Z}[X, Y]$ the irreducible binary form associated to $f_\varepsilon(X)$ via the condition $F_\varepsilon(X, 1) = f_\varepsilon(X)$.

A particular case of Corollary 3.6 of [1] (dealing with Thue–Mahler equations, though Thue equations are involved in this paper), is the following one.

Theorem 1. *Let m be a rational positive integer. Then the set \mathcal{E} , where*

$$\mathcal{E} = \{(x, y, \varepsilon) \in \mathbf{Z}^2 \times \mathbf{Z}_K^\times \mid xy \neq 0, \mathbf{Q}(\alpha\varepsilon) = K \text{ and } |F_\varepsilon(x, y)| \leq m\},$$

is finite.

The proof in [1], involving Schmidt’s subspace theorem, allows to give an effectively computable upper bound for the number of elements of \mathcal{E} as a function of m , d and the absolute logarithmic height $h(\alpha)$ of α (whose definition will be reminded below), but does not allow to give a bound for the heights of the elements of \mathcal{E} . The main result of this paper gives an effectively computable upper bound for the solutions in the particular case where the algebraic number field K has at most one real embedding into \mathbf{C} .

Theorem 2. *Suppose that the algebraic number field K has at most one real embedding. Then there exists an effectively computable constant $\kappa_1 > 0$, depending only on α , such that, for all $m \geq 2$, each solution $(x, y, \varepsilon) \in \mathbf{Z}^2 \times \mathbf{Z}_K^\times$ of the Thue inequality $|F_\varepsilon(x, y)| \leq m$ with $xy \neq 0$ and $\mathbf{Q}(\alpha\varepsilon) = K$ satisfies*

$$\max\{|x|, |y|, e^{h(\alpha\varepsilon)}\} \leq m^{\kappa_1}.$$

The conclusion of Theorem 2 can be stated with the use of the norm $N_{K/\mathbf{Q}}$, since $N_{K/\mathbf{Q}}(X - \alpha\varepsilon Y) = a_0^{-1}F_\varepsilon(X, Y)$: for $(x, y, \varepsilon) \in \mathbf{Z}^2 \times \mathbf{Z}_K^\times$ with $xy \neq 0$ and $\mathbf{Q}(\alpha\varepsilon) = K$, one has

$$|N_{K/\mathbf{Q}}(x - \alpha\varepsilon y)| \geq \kappa_2 \max\{|x|, |y|, e^{h(\alpha\varepsilon)}\}^{\kappa_3}$$

with two effectively computable positive constants κ_2 and κ_3 depending only on α .

If the algebraic number field K has a unique real embedding into \mathbf{C} , the degree d of K is odd and K has $(d-1)/2$ pairs of complex embeddings. In the particular case $d = 3$, namely for a cubic field K , the hypothesis that there is only one real embedding boils down to requiring that the unit rank of K be 1, and this particular case of Theorem 2 was obtained in [2].

When K has no real embedding, namely when K is totally imaginary, the degree d of K is even and K has $d/2$ pairs of complex embeddings. It looks like the totally imaginary case of Theorem 2 is not trivial and requires a diophantine argument. Only the case of a Thue equation is elementary (and not a family of such equations).

Lemma 3. *Let $F(X, Y) \in \mathbf{Z}[X, Y]$ be a binary form of degree d with integer coefficients and without a real root, (so F is a product over \mathbf{R} of definite positive quadratic forms). Then there exists an effectively computable constant κ_4 such that, for each $m > 0$, each solution $(x, y) \in \mathbf{Z}^2$ of the Thue inequality*

$$|F(x, y)| \leq m$$

satisfies

$$\max\{|x|, |y|\} \leq \kappa_4 m^{1/d}.$$

Proof. Write

$$F(X, Y) = a_0 \prod_{j=1}^n (X - \alpha_j Y)(X - \overline{\alpha_j} Y),$$

where $n = d/2$, where a_0 is the leading coefficient of $F(X, 1)$, while α_j and $\overline{\alpha_j}$ ($j = 1, \dots, n$) are the roots of $F(X, 1)$. Since

$$|x - \alpha_j y| = |x - \overline{\alpha_j} y| \geq |\operatorname{Im}(\alpha_j)| \cdot |y|,$$

where $\operatorname{Im}(z)$ is the imaginary part $(z - \bar{z})/2i$ of the complex number z , we have

$$|y| \leq \kappa_5 m^{1/d} \quad \text{with} \quad \kappa_5 = |a_0|^{-1/d} \prod_{j=1}^n |\operatorname{Im}(\alpha_j)|^{-2/d}.$$

If

$$|x| \leq 2 \left(\max_{1 \leq j \leq n} |\alpha_j| \right) |y|,$$

the proof of Lemma 3 is complete. Otherwise, we have $|x - \alpha_j y| \geq |x|/2$ for $j = 1, \dots, n$, and we get

$$|x| \leq 2 \left(\frac{m}{|a_0|} \right)^{1/d},$$

which concludes the proof. \square

The hypothesis that there is at most one real embedding will play an essential role in the proof of Theorem 2. However, a significant part of the proof is valid without this hypothesis, and we plan to pursue our work on the subject.

Example. Let $D \geq 2$, $c \in \{1, -1\}$ and let $K = \mathbf{Q}(\theta)$ with $\theta = (1+i)\sqrt[4]{D^4+c}$. It is a result of Stender [4] that

$$\alpha = \varepsilon = D^2 + D\theta + \frac{1}{2}\theta^2$$

is the fundamental unit of K . Notice that $\theta^4 = -4(D^4 + c)$ and $|\varepsilon| > 1$. The minimal polynomial of ε is

$$f(X) = X^4 - 4D^2X^3 + (8D^4 + 2c)X^2 + 4cD^2X + 1$$

with

$$f(X) = (X - \varepsilon_1)(X - \varepsilon_2)(X - \varepsilon_3)(X - \varepsilon_4)$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ are the conjugates of $\alpha = \varepsilon_1$. Since $X^4 f(-c/X) = f(X)$, we can choose $\varepsilon_3 = -c\varepsilon_1^{-1}$ and $\varepsilon_4 = -c\varepsilon_2^{-1}$. For $n \in \mathbf{Z}$, define $f_n(X)$, a_n , b_n , c_n by

$$X^4 + a_n X^3 + b_n X^2 + c_n X + 1 = (X - \varepsilon_1^n)(X - \varepsilon_2^n)(X - \varepsilon_3^n)(X - \varepsilon_4^n)$$

and

$$f_n(X) = (X - \varepsilon_1^{n+1})(X - \varepsilon_2^{n+1})(X - \varepsilon_3^{n+1})(X - \varepsilon_4^{n+1}),$$

so that

$$f_n(X) = X^4 + a_{n+1}X^3 + b_{n+1}X^2 + c_{n+1}X + 1.$$

Using once more the relation $X^4 f(-c/X) = f(X)$, we deduce

$$c_n = a_{-n} = (-c)^n a_n \quad \text{and} \quad b_{-n} = b_n$$

for $n \in \mathbf{Z}$. The sequence $(a_n)_{n \in \mathbf{Z}}$ satisfies the linear recurrence equation

$$a_{n+4} - 4D^2 a_{n+3} + (8D^4 + 2c)a_{n+2} + 4cD^2 a_{n+1} + a_n = 0$$

with the initial conditions $a_{-1} = 4cD^2$, $a_0 = -4$, $a_1 = -4D^2$, $a_2 = 4c$. The sequence $(c_n)_{n \in \mathbf{Z}}$ is given by $c_n = a_{-n}$ for $n \in \mathbf{Z}$, while the sequence $(b_n)_{n \in \mathbf{Z}}$ satisfies the linear recurrence equation

$$\begin{aligned} & b_{n+6} - (8D^4 + 2c)b_{n+5} - (16cD^4 + 1)b_{n+4} \\ & - (16D^4 - 4c)b_{n+3} - (16cD^4 + 1)b_{n+2} - (8D^4 + 2c)b_{n+1} + b_n = 0, \end{aligned}$$

with the initial conditions

$$\begin{cases} b_{-2} &= 64D^8 + 64cD^4 + 6, \\ b_{-1} &= 8D^4 + 2c, \\ b_0 &= 6, \\ b_1 &= 8D^4 + 2c, \\ b_2 &= 64D^8 + 64cD^4 + 6, \\ b_3 &= 512D^{12} + 1768D^8c + 264D^4 + 2c. \end{cases}$$

Set

$$F_n(X, Y) = Y^4 f_n(X/Y),$$

so that

$$F_n(X, Y) = (X - \varepsilon_1^{n+1}Y)(X - \varepsilon_2^{n+1}Y)(X - \varepsilon_3^{n+1}Y)(X - \varepsilon_4^{n+1}Y).$$

Let m be a rational positive integer. Denote by

$$\mathcal{E} = \{(x, y, n) \in \mathbf{Z}^3 \mid xy \neq 0, n \neq -1 \text{ and } |F_n(x, y)| \leq m\}$$

the set of solutions of the Thue inequality $|F_n(X, Y)| \leq m$. Then from Theorem 2 we deduce the upper bound

$$\max\{|x|, |y|, |\varepsilon|^{|n|} \mid (x, y, n) \in \mathcal{E}\} \leq \kappa_6 m^{\kappa_7},$$

where κ_6 and κ_7 are positive constants depending only on D .

Let us denote by $\Phi = \{\sigma_1, \dots, \sigma_d\}$ the set of embeddings of K into \mathbf{C} . Let us write the irreducible polynomial f of α over \mathbf{Z} as

$$f(X) = a_0X^d + a_1X^{d-1} + \dots + a_d \in \mathbf{Z}[X],$$

so

$$f(X) = a_0 \prod_{i=1}^d (X - \sigma_i(\alpha)),$$

and the irreducible binary form associated to $F \in \mathbf{Z}[X, Y]$ is

$$F(X, Y) = Y^d f(X/Y) = a_0X^d + a_1X^{d-1}Y + \dots + a_dY^d.$$

For $\varepsilon \in \mathbf{Z}_K^\times$ satisfying $\mathbf{Q}(\alpha\varepsilon) = K$, we have

$$F_\varepsilon(X, Y) = a_0 \prod_{i=1}^d (X - \sigma_i(\alpha\varepsilon)Y) \in \mathbf{Z}[X, Y].$$

Let us denote by h the absolute logarithmic height and by M the Mahler measure. Hence,

$$h(\alpha) = \frac{1}{d} \log M(\alpha) \quad \text{where} \quad M(\alpha) = a_0 \prod_{1 \leq i \leq d} \max\{1, |\sigma_i(\alpha)|\}.$$

2 Tools

In this section, let us put together the auxiliary lemmas which will prove useful. We will use some results in geometry of numbers to establish an equivalence of norms (Lemma 5). Then we state in Lemma 6 what is Lemma 2 of [2]. Finally in Proposition 8 and in Corollaries 9 and 10 we exhibit some lower bounds of linear forms of logarithms of algebraic numbers.

2.1 Equivalence of norms

Let K be an algebraic number field of degree d . We denote by $\epsilon_1, \dots, \epsilon_r$ a basis of the unit group of K modulo K_{tors}^\times and we suppose $r \geq 1$. Let us recall that the house of the algebraic number γ , denoted $|\overline{\gamma}|$, is the maximum of the absolute values of the conjugates of γ .

Let us first remark that there exists a positive constant κ_8 , depending only on $\epsilon_1, \dots, \epsilon_r$, such that, if c_1, \dots, c_r are rational integers and if we let

$$\gamma = \epsilon_1^{c_1} \cdots \epsilon_r^{c_r} \quad \text{with} \quad C = \max\{|c_1|, \dots, |c_r|\}$$

be a unit of K , then

$$e^{-\kappa_8 C} \leq |\varphi(\gamma)| \leq e^{\kappa_8 C} \tag{4}$$

for any embedding φ of K into \mathbf{C} . More precisely, the inequalities

$$|\varphi(\gamma)| \leq \prod_{i=1}^r |\overline{\epsilon_i}|^C \quad \text{and} \quad |\varphi(\gamma^{-1})| \leq \prod_{i=1}^r |\overline{\epsilon_i}|^C$$

(note that $|\overline{\epsilon_i}| \geq 1$) suggest to take

$$\kappa_8 = \sum_{i=1}^r \log |\overline{\epsilon_i}|.$$

The following result, Lemma 5, is a variant of Lemma 5.1 of [3]. It shows that the two inequalities of (4) are optimal. They make C appear as an upper bound, while the two inequalities of the conclusion of Lemma 5 make C appear as a lower bound.

Lemma 5. *There exists an effectively computable positive constant κ_9 , depending only on $\epsilon_1, \dots, \epsilon_r$, with the following property. If c_1, \dots, c_r are rational integers and if we let*

$$\gamma = \epsilon_1^{c_1} \cdots \epsilon_r^{c_r}, \quad C = \max\{|c_1|, \dots, |c_r|\},$$

then there exist two embeddings σ and τ of K into \mathbf{C} such that

$$|\sigma(\gamma)| \geq e^{\kappa_9 C} \quad \text{and} \quad |\tau(\gamma)| \leq e^{-\kappa_9 C}.$$

Obviously, $\kappa_9 \leq \kappa_8$.

Proof. For any embedding φ of K into \mathbf{C} , we have

$$\log |\varphi(\gamma)| = c_1 \log |\varphi(\epsilon_1)| + \cdots + c_r \log |\varphi(\epsilon_r)|.$$

The matrix $(\log |\varphi(\epsilon_i)|)$, with d lines and r columns, where the indices for the lines are the φ 's of Φ and the indices for the columns are the j 's of the set $\{1, 2, \dots, r\}$, has rank r . Therefore we can write c_1, \dots, c_r as linear combinations of the elements $\log |\varphi(\gamma)|$, whose coefficients in absolute values have upper bounds which are functions of the regulator of K . Hence

$$\max\{|c_1|, \dots, |c_r|\} \leq \kappa_{10} \max_{\varphi \in \Phi} (\log |\varphi(\gamma)|).$$

Let us take for σ an element of Φ such that $|\sigma(\gamma)|$ be maximal:

$$|\sigma(\gamma)| = \max_{\varphi \in \Phi} |\varphi(\gamma)|.$$

This leads to

$$C \leq \kappa_{10} \log |\sigma(\gamma)|,$$

providing the conclusion for σ with $\kappa_9 = \kappa_{10}^{-1}$.

We next apply this result to

$$\gamma^{-1} = \epsilon_1^{-c_1} \cdots \epsilon_r^{-c_r}.$$

If τ is an element of Φ such that $|\tau(\gamma)|$ be minimal, namely

$$|\tau(\gamma)| = \min_{\varphi \in \Phi} |\varphi(\gamma)|,$$

then we have

$$C \leq \kappa_{10} \log |\tau(\gamma^{-1})|.$$

□

Remark. Under the hypotheses of Lemma 5, if γ_0 is a nonzero element of K and if we let $\gamma_1 = \gamma_0 \gamma$, we deduce

$$e^{-\kappa_8 C} \leq \min_{\varphi \in \Phi} |\varphi(|\gamma_0|^{-1}) \varphi(\gamma_1)| \leq e^{-\kappa_9 C}$$

and

$$e^{\kappa_9 C} \leq \max_{\varphi \in \Phi} |\varphi(|\gamma_0|^{-1}) \varphi(\gamma_1)| \leq e^{\kappa_8 C}.$$

To benefit from these inequalities, we will use the estimates

$$|\varphi(\gamma_0)| \leq |\overline{\gamma_0}| \leq e^{dh(\gamma_0)} \quad \text{and} \quad h(\gamma_0^{-1}) = h(\gamma_0)$$

from which we deduce

$$e^{-\kappa_8 C - dh(\gamma_0)} \leq \min_{\varphi \in \Phi} |\varphi(\gamma_1)| \leq e^{-\kappa_9 C + dh(\gamma_0)}$$

and

$$e^{\kappa_9 C - dh(\gamma_0)} \leq \max_{\varphi \in \Phi} |\varphi(\gamma_1)| \leq e^{\kappa_8 C + dh(\gamma_0)}.$$

The term $dh(\gamma_0)$ always appears as an upper bound, and as an error term. Note also that the constant κ_8 is large (it appears in the upper bounds), while the constant κ_9 is small (it comes into play in some lower bounds).

2.2 Upper bound involving the norm

Given an algebraic number γ of degree $\leq d$ and norm $\leq m$, there exists a unit ε in the field $\mathbf{Q}(\alpha)$ such that the conjugates of $\varepsilon\gamma$ are bounded from above by a constant times $m^{1/d}$. This is a consequence of Lemma A.15 of [3], a result which we want to state; (this is also Lemma 2 of [2]).

Lemma 6. *Let K be an algebraic number field of degree d with regulator R and let γ be a nonzero element of \mathbf{Z}_K whose norm in absolute value is m . Then there exists a unit $\varepsilon \in \mathbf{Z}_K^\times$ such that*

$$\frac{1}{R} \max_{1 \leq j \leq d} \left| \log(m^{-1/d} |\sigma_j(\varepsilon\gamma)|) \right|$$

is bounded from above by an effectively computable constant depending only on d .

Remark. The unit $\varepsilon \in \mathbf{Z}_K^\times$ may come from the group generated by the basis of the unit group modulo the torsion. In other words, the torsion elements do not come into play because of the absolute values which appear in $|\sigma_j(\varepsilon\gamma)|$.

We use this lemma in the same way we did in [2]: there exists an effectively computable constant κ_{11} , which is a function of d and R , such that, if γ is a nonzero element of \mathbf{Z}_K whose norm has an absolute value $\leq m$, then there exists a unit $\varepsilon \in \mathbf{Z}_K^\times$ such that

$$\max_{1 \leq j \leq d} |\sigma_j(\varepsilon\gamma)| \leq \kappa_{11} m^{1/d}.$$

We will suppose $m \geq 2$ and we will rather use the weaker upper bound

$$\max_{1 \leq j \leq d} |\sigma_j(\varepsilon\gamma)| \leq m^{\kappa_{12}} \quad (7)$$

with κ_{12} , an effectively computable constant which can be calculated as a function of d and R .

2.3 Diophantine tools

In this section, we are given two positive integers s and D ; the constants κ_{13} , κ_{14} and κ_{15} depend only on s and D and are effectively computable.

Here are the hypotheses and notations common to Proposition 8 and to Corollaries 9 and 10. Let $\gamma_1, \dots, \gamma_s$ be algebraic numbers in an algebraic number field of degree $\leq D$. Let c_1, \dots, c_s be rational integers. Suppose $\gamma_1^{c_1} \cdots \gamma_s^{c_s} \neq 1$. Moreover, let H_1, \dots, H_s be real numbers ≥ 1 which satisfy

$$H_j \geq h(\gamma_j) \quad (1 \leq j \leq s).$$

We will use the following particular case of Theorem 9.1 of [5] (see Proposition 9.21 of [5]).

Proposition 8. *Let $\lambda_1, \dots, \lambda_s$ be complex numbers such that $\gamma_j = e^{\lambda_j}$ for all $j \in \{1, \dots, s\}$. Put*

$$\Lambda = c_1 \lambda_1 + \dots + c_s \lambda_s.$$

Suppose $H_j \geq e^{|\lambda_j|}$ for $j = 1, \dots, s$. Let C_0 be a real number satisfying

$$C_0 \geq 2, \quad C_0 \geq \max_{1 \leq j < s} \left\{ \frac{|c_s|}{H_j} + \frac{|c_j|}{H_s} \right\}.$$

Then

$$|\Lambda| \geq \exp\{-\kappa_{13} H_1 \cdots H_s \log C_0\}.$$

Note that the hypothesis $\gamma_1^{c_1} \cdots \gamma_s^{c_s} \neq 1$ implies $\Lambda \neq 0$. Proposition 8 will be used via the following corollary.

Corollary 9. *Suppose $H_j \leq H_s$ for $1 \leq j \leq s$. Let C_1 a real number satisfying*

$$C_1 \geq 2, \quad C_1 \geq \max_{1 \leq j \leq s} \left\{ \frac{H_j}{H_s} |c_j| \right\}.$$

Then

$$|\gamma_1^{c_1} \cdots \gamma_s^{c_s} - 1| > \exp\{-\kappa_{14} H_1 \cdots H_s \log C_1\}.$$

Proof. Under the hypotheses of Corollary 9, we choose an embedding of the algebraic number field $K = \mathbf{Q}(\gamma_1, \dots, \gamma_s)$ into \mathbf{C} . From the definition of the absolute logarithmic height h , we deduce $|\gamma_j| \leq e^{Dh(\gamma_j)}$ for $j = 1, \dots, s$. We introduce some real numbers ν_1, \dots, ν_s via the conditions

$$\gamma_j = |\gamma_j| e^{2i\pi\nu_j}, \quad -1 < \nu_j \leq 1 \quad (1 \leq j \leq s)$$

and we put

$$\lambda_j = \log |\gamma_j| + 2i\pi\nu_j \quad (1 \leq j \leq s).$$

In other words, λ_j is the main determination of the logarithm of γ_j . In particular, $e^{\lambda_j} = \gamma_j$ and $|\lambda_j| \leq DH_j + 2\pi$ ($1 \leq j \leq s$). Write

$$\Lambda_0 = c_1 \lambda_1 + \dots + c_s \lambda_s.$$

So

$$e^{\Lambda_0} = \gamma_1^{c_1} \cdots \gamma_s^{c_s} \neq 1.$$

Note also Λ the main determination of the logarithm of $\gamma_1^{c_1} \cdots \gamma_s^{c_s}$. So there exists $c_0 \in \mathbf{Z}$ such that $\Lambda = 2i\pi c_0 + \Lambda_0$. We may suppose $|e^\Lambda - 1| < 1/2$; otherwise, the conclusion of Corollary 9 is trivial; we deduce (for instance, see Lemma 3 of [2])

$$|\Lambda| \leq 2|\gamma_1^{c_1} \cdots \gamma_s^{c_s} - 1|.$$

In the same vein, write $\lambda_0 = 2i\pi$, $\gamma_0 = 1$, $H_0 = 2\pi$, so

$$\Lambda = c_0 \lambda_0 + c_1 \lambda_1 + \dots + c_s \lambda_s.$$

From the upper bound $|\Lambda| \leq 1$, we deduce

$$2\pi|c_0| \leq 1 + \sum_{j=1}^s |c_j \lambda_j| \leq 1 + \sum_{j=1}^s (DH_j + 2\pi)|c_j|.$$

Therefore the inequalities

$$C_1 \geq \frac{H_j}{H_s} |c_j|, \quad H_j \geq 1 \quad (1 \leq j \leq s)$$

and

$$\frac{|c_0|}{H_s} \leq \left(\frac{1}{2\pi} + \frac{sD}{2\pi} + s \right) C_1 \leq 2sDC_1$$

allow us to use Proposition 8 with s replaced by $s+1$ and $4sDC_1$ playing the role of C_0 . This concludes the proof. \square

The following particular case of Corollary 9 is also deduced from Corollary 9.22 of [5].

Corollary 10. *Let C_2 be a real number satisfying*

$$C_2 \geq \max\{2, |c_1|, \dots, |c_s|\}.$$

Then

$$|\gamma_1^{c_1} \cdots \gamma_s^{c_s} - 1| > \exp\{-\kappa_{15} H_1 \cdots H_s \log C_2\}.$$

3 The reciprocal polynomial

Let us show that there is no restriction, for the statement of Theorem 2, to suppose $|x| \leq |y|$. Suppose that Theorem 2 is proved in the particular case $|x| \leq |y|$ with the constant κ_1 replaced by $\kappa_{16}(\alpha)$. Suppose now that we are under the hypotheses of Theorem 2 and consider a solution $(x, y, \varepsilon) \in \mathbf{Z}^2 \times \mathbf{Z}_K^\times$ of the inequality $|F_\varepsilon(x, y)| \leq m$ with $xy \neq 0$ and $|x| > |y|$. Write $\alpha' = \alpha^{-1}$; let $g \in \mathbf{Z}[Y]$ be the irreducible polynomial of α' :

$$g(Y) = Y^d f(Y^{-1}) = a_d \prod_{i=1}^d (Y - \sigma_i(\alpha')).$$

Put

$$\varepsilon' = \varepsilon^{-1}, \quad g_{\varepsilon'}(Y) = a_d \prod_{i=1}^d (Y - \sigma_i(\alpha' \varepsilon')), \quad G_{\varepsilon'}(Y, X) = X^d g_{\varepsilon'}(Y/X),$$

so that

$$G_{\varepsilon'}(Y, X) = F_\varepsilon(X, Y).$$

Consequently, (y, x, ε') is a solution of the Thue inequality

$$|G_{\varepsilon'}(y, x)| \leq m.$$

Since $h((\alpha\varepsilon)^{-1}) = h(\alpha\varepsilon)$, we deduce from Theorem 2 applied to G_ε , with the condition $|y| < |x|$,

$$\max\{|x|, e^{h(\alpha\varepsilon)}\} \leq m^{\kappa_{16}(\alpha^{-1})},$$

which allows us to draw the conclusion, by taking $\kappa_1 = \max\{\kappa_{16}(\alpha), \kappa_{16}(\alpha^{-1})\}$.

From now on, we fix an element $(x, y, \varepsilon) \in \mathcal{E}$ with $|x| \leq |y|$. The constants $\kappa_{17}, \kappa_{18}, \dots, \kappa_{63}$ which will appear, depend only on $h(\alpha)$, d and m ; they are easy to calculate explicitly.

4 Introduction of the parameters \tilde{A} , A , \tilde{B} , B

Put

$$\tilde{A} = \max\{1, h(\alpha\varepsilon)\}.$$

Write

$$\varepsilon = \zeta \epsilon_1^{a_1} \cdots \epsilon_r^{a_r}$$

with $\zeta \in K_{\text{tors}}^\times$ and $a_i \in \mathbf{Z}$ for $1 \leq i \leq r$, and put

$$A = \max\{1, |a_1|, \dots, |a_r|\}.$$

The upper bound

$$h(\alpha\varepsilon) \leq \kappa_{17}A$$

follows from (4); more precisely, we may take

$$\kappa_{17} = \max\{1, h(\alpha) + h(\epsilon_1) + \cdots + h(\epsilon_r)\}.$$

The lower bound

$$h(\alpha\varepsilon) \geq \kappa_{18}A$$

follows from Lemma 5. Consequently,

$$\kappa_{18}A \leq \tilde{A} \leq \kappa_{17}A.$$

Next put

$$\beta = x - \alpha\varepsilon y \quad \text{and} \quad \tilde{B} = \max\{1, h(\beta)\}.$$

So

$$F_\varepsilon(x, y) = a_0 \sigma_1(\beta) \cdots \sigma_d(\beta). \tag{11}$$

Since $|F_\varepsilon(x, y)| \leq m$, it follows from (7), (11) and Lemma 6 that there exists $\varrho \in \mathbf{Z}_K$ satisfying $h(\varrho) \leq \kappa_{19} \log m$ and such that $\eta = \beta/\varrho$ belongs to the subgroup of the unit group of \mathbf{Z}_K generated by $\epsilon_1, \dots, \epsilon_r$. Write

$$\eta = \epsilon_1^{b_1} \cdots \epsilon_r^{b_r}$$

with rational numbers b_1, \dots, b_r and put

$$B = \max\{1, |b_1|, |b_2|, \dots, |b_r|\}.$$

With the relation $\beta = \varrho\eta$, we deduce from (4),

$$\tilde{B} \leq \kappa_{20}(B + \log m)$$

and from Lemma 5 we get

$$B \leq \kappa_{21}(\tilde{B} + \log m).$$

By using the hypothesis $xy \neq 0$, we verify that the condition $\mathbf{Q}(\alpha\varepsilon) = K$ which appears in the definition of \mathcal{E} implies $\mathbf{Q}(\beta) = \mathbf{Q}(\beta/\alpha\varepsilon) = K$. Consequently, for φ and σ in Φ , we have

$$\varphi = \sigma \iff \varphi(\alpha\varepsilon) = \sigma(\alpha\varepsilon) \iff \varphi(\beta) = \sigma(\beta) \iff \sigma(\alpha\varepsilon)\varphi(\beta) = \sigma(\beta)\varphi(\alpha\varepsilon).$$

5 Elimination

5.1 Expressions of x and y in terms of $\alpha\varepsilon$ and β

Let φ_1, φ_2 be two distinct elements of Φ , *i.e.*, two distinct embeddings of K into \mathbf{C} . Let us eliminate x (resp. y) between the two equations

$$\varphi_1(\beta) = x - \varphi_1(\alpha\varepsilon)y \quad \text{and} \quad \varphi_2(\beta) = x - \varphi_2(\alpha\varepsilon)y.$$

This leads to

$$y = \frac{\varphi_1(\beta) - \varphi_2(\beta)}{\varphi_2(\alpha\varepsilon) - \varphi_1(\alpha\varepsilon)}, \quad x = \frac{\varphi_1(\beta)\varphi_2(\alpha\varepsilon) - \varphi_2(\beta)\varphi_1(\alpha\varepsilon)}{\varphi_2(\alpha\varepsilon) - \varphi_1(\alpha\varepsilon)}. \quad (12)$$

5.2 A Siegel unit equation

Let $\varphi_1, \varphi_2, \varphi_3$ be three elements of Φ , *i.e.*, three embeddings of K into \mathbf{C} . Write

$$u_i = \varphi_i(\alpha\varepsilon), \quad v_i = \varphi_i(\beta) \quad (i = 1, 2, 3).$$

We can eliminate x and y between the three equations

$$\begin{cases} \varphi_1(\beta) &= x - \varphi_1(\alpha\varepsilon)y \\ \varphi_2(\beta) &= x - \varphi_2(\alpha\varepsilon)y \\ \varphi_3(\beta) &= x - \varphi_3(\alpha\varepsilon)y \end{cases}$$

by writing that the determinant

$$\begin{vmatrix} 1 & \varphi_1(\alpha\varepsilon) & \varphi_1(\beta) \\ 1 & \varphi_2(\alpha\varepsilon) & \varphi_2(\beta) \\ 1 & \varphi_3(\alpha\varepsilon) & \varphi_3(\beta) \end{vmatrix} = \begin{vmatrix} 1 & u_1 & v_1 \\ 1 & u_2 & v_2 \\ 1 & u_3 & v_3 \end{vmatrix}$$

is equal to 0, and this leads to the unit equation *à la Siegel*

$$u_1v_2 - u_1v_3 + u_2v_3 - u_2v_1 + u_3v_1 - u_3v_2 = 0. \quad (13)$$

6 Four privileged embeddings

Let us denote by σ_a (resp. σ_b) an embedding φ of K into \mathbf{C} having the property that $|\varphi(\alpha\varepsilon)|$ (resp. $|\varphi(\beta)|$) be maximal among all the elements $|\varphi(\alpha\varepsilon)|$ (resp. $|\varphi(\beta)|$) when φ runs through all the embeddings of Φ . Let us denote by τ_a (resp. τ_b) an embedding ψ of K into \mathbf{C} having the property that $|\psi(\alpha\varepsilon)|$ (resp. $|\psi(\beta)|$) be minimal among all the elements $|\psi(\alpha\varepsilon)|$ (resp. $|\psi(\beta)|$) when ψ runs through all the embeddings of Φ . We may suppose $\tau_b \neq \sigma_b$, $\tau_b \neq \overline{\sigma_b}$, together with $\tau_a \neq \sigma_a$, $\tau_a \neq \overline{\sigma_a}$.

These four embeddings will be used in the unit equation in many different ways.

7 About the parameters A and B

It turns out that if we are under the hypothesis $\max\{A, B\} \leq \kappa \log m$, we can easily prove Theorem 2.

7.1 Proof of Theorem 2 if $\max\{A, B\} \leq \kappa \log m$

We indeed have a short proof of Theorem 2 in this case.

Lemma 14. *If κ is a real number such that $\max\{A, B\} \leq \kappa \log m$, then the conclusion of Theorem 2 holds true with a constant κ_1 depending not only on α but also on κ .*

Proof. In this proof, the constants κ_{22} , κ_{23} , κ_{24} , κ_{25} , κ_{26} depend not only on α but also on κ . We use (12) with the estimates

$$|\varphi_1(\beta) - \varphi_2(\beta)| \leq e^{\kappa_{22}B}, \quad |\varphi_2(\alpha\varepsilon) - \varphi_1(\alpha\varepsilon)| \geq e^{-\kappa_{23}A},$$

and the estimates

$$|\varphi_1(\beta)\varphi_2(\alpha\varepsilon) - \varphi_2(\beta)\varphi_1(\alpha\varepsilon)| \leq e^{\kappa_{24}(A+B)},$$

to successively obtain the upper bounds $|y| \leq m^{\kappa_{25}}$ and then $|x| \leq m^{\kappa_{26}}$. Therefore it follows that, if an upper bound of both A and B is $\kappa \log m$, the proof of Theorem 2 is secured. \square

The goal is now to show that we can assume that we always are under the hypothesis $\max\{A, B\} \leq \kappa \log m$. In the next two subsections, we show that this goal is achieved if we assume that an upper bound of either A or B is given by a constant times $\log m$. This will mean that there is no restriction in supposing A and B larger than $\kappa_{27} \log m$ with a constant κ_{27} we may assume to be sufficiently large. Once more, we could produce a convenient explicit value for this constant which will make valid the arguments which will be used.

7.2 Proof of Theorem 2 if $A \leq \kappa' \log m$

In the next lemma, we prove that if $A \leq \kappa' \log m$, then we have the inequality $\max\{A, B\} \leq \kappa \log m$, whereupon Lemma 14 states that Theorem 2 holds true.

Lemma 15. *If $\kappa' > 0$ is a constant such that $A \leq \kappa' \log m$, then the hypothesis $\max\{A, B\} \leq \kappa \log m$ of Lemma 14 holds true with a constant κ depending upon α and κ' .*

Proof. In this proof, the constants $\kappa_{28}, \kappa_{29}, \kappa_{30}, \kappa_{31}, \kappa_{32}$ depend not only on α but also on κ' . Let us use the unit equation (13) with the two embeddings σ_b, τ_b and a third embedding φ distinct from σ_b and τ_b :

$$\frac{\varphi(\beta)}{\sigma_b(\beta)} \cdot \frac{\sigma_b(\alpha\varepsilon) - \tau_b(\alpha\varepsilon)}{\varphi(\alpha\varepsilon) - \tau_b(\alpha\varepsilon)} - 1 = -\frac{\tau_b(\beta)}{\sigma_b(\beta)} \cdot \frac{\varphi(\alpha\varepsilon) - \sigma_b(\alpha\varepsilon)}{\varphi(\alpha\varepsilon) - \tau_b(\alpha\varepsilon)}. \quad (16)$$

The member on the right side is nonzero, since $\varphi \neq \sigma_b$; it has a modulus $\leq e^{-\kappa_{28}B} e^{\kappa_{29}A}$, since

$$|\tau_b(\beta)| \leq e^{-\kappa_{39}B}, \quad |\sigma_b(\beta)| \geq e^{\kappa_{39}B}$$

and

$$\left| \frac{\varphi(\alpha\varepsilon) - \sigma_b(\alpha\varepsilon)}{\varphi(\alpha\varepsilon) - \tau_b(\alpha\varepsilon)} \right| \leq e^{\kappa_{29}A}.$$

Put $s = r + 1$, and

$$\gamma_j = \frac{\varphi(\epsilon_j)}{\sigma_b(\epsilon_j)}, \quad c_j = b_j \quad (1 \leq j \leq r), \quad \gamma_s = \frac{\varphi(\varrho)}{\sigma_b(\varrho)} \cdot \frac{\sigma_b(\alpha\varepsilon) - \tau_b(\alpha\varepsilon)}{\varphi(\alpha\varepsilon) - \tau_b(\alpha\varepsilon)}, \quad c_s = 1,$$

so

$$\gamma_1^{c_1} \cdots \gamma_s^{c_s} = \frac{\varphi(\beta)}{\sigma_b(\beta)} \cdot \frac{\sigma_b(\alpha\varepsilon) - \tau_b(\alpha\varepsilon)}{\varphi(\alpha\varepsilon) - \tau_b(\alpha\varepsilon)}.$$

Put

$$H_1 = \cdots = H_r = \kappa_{30}, \quad H_s = \kappa_{30}(1 + \log m), \quad C_1 = 2 + \frac{B}{1 + \log m}.$$

For $1 \leq j \leq r$, we have $|c_j| \leq B$ and

$$\frac{H_j}{H_s} |c_j| \leq \kappa_{30} \frac{B}{H_s} \leq \frac{B}{1 + \log m} \leq C_1.$$

The upper bound

$$\frac{H_j}{H_s} |c_j| \leq C_1$$

is still valid for $j = s$ since $c_s = 1$. Corollary 9 shows that a lower bound of the left member of (16) is given by $\exp\{-\kappa_{31}H_s \log C_1\}$, whereupon the above upper bound and the last lower bound lead to

$$\exp\{-\kappa_{31}H_s \log C_1\} \leq \exp\{-\kappa_{28}B + \kappa_{29}A\}.$$

Therefore

$$\kappa_{28}B \leq \kappa_{31}H_s \log C_1 + \kappa_{29}A,$$

which shows that $B \leq \kappa_{32} \log m$, which is what we wanted to establish. \square

From now on, without loss of generality we may assume that A is sufficiently large, say, $A \geq \kappa_{27} \log m$.

7.3 Proof of Theorem 2 if $B \leq \kappa'' \log m$

In the next lemma, we prove that if $B \leq \kappa'' \log m$, then we have the inequality $\max\{A, B\} \leq \kappa \log m$, whereupon Lemma 14 states that Theorem 2 holds true.

Lemma 17. *If $\kappa'' > 0$ is a constant such that $B \leq \kappa'' \log m$, then the hypothesis $\max\{A, B\} \leq \kappa \log m$ of Lemma 14 holds true with a constant κ depending on α and κ'' .*

Proof. In this proof, the constants $\kappa_{33}, \kappa_{34}, \kappa_{35}, \kappa_{36}, \kappa_{37}$ depend not only on α but also on κ'' .

Assume $B \leq \kappa'' \log m$. Let us use the unit equation (13) with the two embeddings σ_a, τ_a and a third one φ distinct from σ_a and τ_a :

$$\frac{\varphi(\alpha\varepsilon)}{\sigma_a(\alpha\varepsilon)} \cdot \frac{\sigma_a(\beta) - \tau_a(\beta)}{\varphi(\beta) - \tau_a(\beta)} - 1 = -\frac{\tau_a(\alpha\varepsilon)}{\sigma_a(\alpha\varepsilon)} \cdot \frac{\varphi(\beta) - \sigma_a(\beta)}{\varphi(\beta) - \tau_a(\beta)}. \quad (18)$$

The member on the right side is nonzero, since $\varphi \neq \sigma_a$; it has a modulus $\leq e^{-\kappa_{33}A} e^{\kappa_{34}B}$ since

$$|\tau_a(\alpha\varepsilon)| \leq e^{-\kappa_{39}A}, \quad |\sigma_a(\alpha\varepsilon)| \geq e^{\kappa_{39}A} \quad \text{and} \quad \left| \frac{\varphi(\beta) - \sigma_a(\beta)}{\varphi(\beta) - \tau_a(\beta)} \right| \leq e^{\kappa_{34}B}.$$

Put $s = r + 1$ and

$$\gamma_j = \frac{\varphi(\epsilon_j)}{\sigma_a(\epsilon_j)}, \quad c_j = a_j \quad (1 \leq j \leq r), \quad \gamma_s = \frac{\varphi(\alpha\zeta)}{\sigma_a(\alpha\zeta)} \cdot \frac{\sigma_a(\beta) - \tau_a(\beta)}{\varphi(\beta) - \tau_a(\beta)}, \quad c_s = 1,$$

so

$$\gamma_1^{c_1} \cdots \gamma_s^{c_s} = \frac{\varphi(\alpha\varepsilon)}{\sigma_a(\alpha\varepsilon)} \cdot \frac{\sigma_a(\beta) - \tau_a(\beta)}{\varphi(\beta) - \tau_a(\beta)}.$$

Now put

$$H_1 = \cdots = H_r = \kappa_{35}, \quad H_s = \kappa_{35}(1 + \log m), \quad C_1 = 2 + \frac{A}{1 + \log m}.$$

Corollary 9 shows that a lower bound of the left member of (18) is given by $\exp\{-\kappa_{36}H_s \log C_1\}$, whereupon the above upper bound and the last lower bound lead to

$$\exp\{-\kappa_{36}H_s \log C_1\} \leq \exp\{-\kappa_{33}A + \kappa_{34}B\}.$$

Therefore

$$\kappa_{33}A \leq \kappa_{34}B + \kappa_{36}H_s \log C_1.$$

We conclude $A \leq \kappa_{37} \log m$, which is what we wanted to achieve. \square

7.4 Consequences

From now on, without loss of generality, we may assume that both A and B are larger than $\kappa_{27} \log m$. Since A is sufficiently large, we have $|\sigma_a(\alpha\varepsilon)| \geq 2$ and $|\tau_a(\alpha\varepsilon)| \leq 1/2$. Similarly, since B is sufficiently large, we have $|\sigma_b(\beta)| \geq 2$ and $|\tau_b(\beta)| \leq 1/2$.

By using Lemma 5 with the bounds (4), we deduce that there exist some constants κ_{38} et κ_{39} such that

$$\begin{cases} e^{\kappa_{39}A} \leq |\sigma_a(\alpha\varepsilon)| \leq e^{\kappa_{38}A}, & e^{-\kappa_{38}A} \leq |\tau_a(\alpha\varepsilon)| \leq e^{-\kappa_{39}A}, \\ e^{\kappa_{39}B} \leq |\sigma_b(\beta)| \leq e^{\kappa_{38}B}, & e^{-\kappa_{38}B} \leq |\tau_b(\beta)| \leq e^{-\kappa_{39}B}. \end{cases} \quad (19)$$

Remark. Note that the constant κ_{39} is sufficiently small (it comes into play as a lower bound) while κ_{38} is sufficiently large (it comes into play as an upper bound).

Lemma 20. *We have*

$$|\tau_b(\alpha\varepsilon)| \leq 2 \quad \text{and} \quad |\sigma_a(\beta)| \geq e^{\kappa_{40}A}.$$

Proof. On the one hand, we have

$$|x - \tau_b(\alpha\varepsilon)y| = |\tau_b(\beta)| \leq 1 \leq |x|,$$

and since $|x| \leq |y|$, we get

$$|\tau_b(\alpha\varepsilon)y| \leq 2|x| \leq 2|y|.$$

On the other hand, upon using $|x| \leq |y|$ and $|\sigma_a(\alpha\varepsilon)| \geq 2$, we find

$$|\sigma_a(\beta)| \geq |\sigma_a(\alpha\varepsilon)y| - |x| \geq (|\sigma_a(\alpha\varepsilon)| - 1)|y| \geq \frac{1}{2}|\sigma_a(\alpha\varepsilon)| \geq e^{\kappa_{40}A}.$$

□

8 An upper bound for A , $|x|$, $|y|$ involving B

From the relation (12), we will deduce in an elementary way the following upper bound.

Lemma 21. *We have*

$$A \leq \kappa_{41}B$$

and

$$\log \max\{|x|, |y|\} \leq \kappa_{42}B.$$

Proof. Since A is sufficiently large, we have $|\sigma_a(\alpha\varepsilon)| \geq 2|\tau_a(\alpha\varepsilon)|$. We then deduce

$$|\sigma_a(\alpha\varepsilon) - \tau_a(\alpha\varepsilon)| \geq \frac{1}{2}|\sigma_a(\alpha\varepsilon)|.$$

Let us use (12) with $\varphi_2 = \sigma_a$ and $\varphi_1 = \tau_a$:

$$y(\sigma_a(\alpha\varepsilon) - \tau_a(\alpha\varepsilon)) = \tau_a(\beta) - \sigma_a(\beta).$$

The very definition of σ_b gives the upper bound

$$|\sigma_a(\beta) - \tau_a(\beta)| \leq 2|\sigma_b(\beta)|,$$

from which we deduce

$$|y\sigma_a(\alpha\varepsilon)| \leq 4|\sigma_b(\beta)|. \quad (22)$$

The inequalities

$$e^{\kappa_{39}A} \leq |\sigma_a(\alpha\varepsilon)| \leq |y\sigma_a(\alpha\varepsilon)| \leq 4|\sigma_b(\beta)| \leq 4e^{\kappa_{38}B}$$

imply $A \leq \kappa_{41}B$. From (22) we deduce

$$|y| \leq 4e^{\kappa_{38}B}.$$

The upper bound for $|x|$ follows from $|x| \leq |y|$. This concludes the proof of Lemma 21. \square

9 On the unicity of τ_b

We plan to show that the embedding τ_b , that we have defined, is unique.

Lemma 23. *Suppose that there exists $\varphi \in \Phi$, $\varphi \neq \tau_b$ such that we have $|\varphi(\beta)| = |\tau_b(\beta)|$. Then $B \leq \kappa_{43}$.*

Proof. Suppose $|\varphi(\beta)| = |\tau_b(\beta)|$ with $\varphi \neq \tau_b$. Let us use (12) with $\varphi_1 = \varphi$, $\varphi_2 = \tau_b$, under the form

$$-\varphi(\alpha\varepsilon) + \tau_b(\alpha\varepsilon) = \frac{1}{|y|}(\varphi(\beta) - \tau_b(\beta)).$$

We have

$$|\varphi(\alpha\varepsilon) - \tau_b(\alpha\varepsilon)| = \frac{1}{|y|}|\varphi(\beta) - \tau_b(\beta)| \leq \frac{2|\tau_b(\beta)|}{|y|}.$$

From

$$|x - \tau_b(\alpha\varepsilon)y| = |\tau_b(\beta)| \leq e^{-\kappa_{39}B} < \frac{1}{2},$$

we deduce

$$|\tau_b(\alpha\varepsilon)y| \geq |x| - \frac{1}{2} \geq \frac{1}{2}.$$

Consequently,

$$\left| \frac{\varphi(\alpha\varepsilon)}{\tau_b(\alpha\varepsilon)} - 1 \right| \leq \frac{2|\tau_b(\beta)|}{|\tau_b(\alpha\varepsilon)y|} \leq 4|\tau_b(\beta)| \leq 4e^{-\kappa_{39}B}.$$

Let us write the member on the left in the form $|\gamma_1^{c_1} \cdots \gamma_s^{c_s} - 1|$ with $s = r + 1$,

$$\gamma_i = \frac{\varphi(\epsilon_i)}{\tau_b(\epsilon_i)}, \quad c_i = a_i, \quad (i = 1, \dots, r), \quad \gamma_s = \frac{\varphi(\alpha\zeta)}{\tau_b(\alpha\zeta)}, \quad c_s = 1.$$

From Corollary 10 with

$$H_1 = \cdots = H_s = \kappa_{44}, \quad C_2 = A,$$

(note that $C_2 \geq 2$ since A is sufficiently large) we deduce

$$|\gamma_1^{c_1} \cdots \gamma_s^{c_s} - 1| \geq e^{-\kappa_{45} \log A}.$$

The above upper bound and the last lower bound lead to

$$e^{-\kappa_{45} \log A} \leq 4e^{-\kappa_{39}B},$$

hence $B \leq \kappa_{46}(\log A)$. Using the upper bound $A \leq \kappa_{41}B$ of Lemma 21 allows us to conclude. \square

As outlined in §8, the upper bound $B \leq \kappa_{43} \log m$ allows to secure the proof of Theorem 2. So Lemma 23 allows us to suppose $|\varphi(\beta)| > |\tau_b(\beta)|$ for any $\varphi \in \Phi$ different from τ_b . In particular, the embedding τ_b is real. This completes the proof in the case where the algebraic number field $\mathbf{Q}(\alpha)$ is totally imaginary.

When the algebraic number field K has a unique real embedding, which is what we will assume from now on, all embeddings different from τ_b are imaginary. In particular, since $\sigma_b \neq \tau_b$ we deduce $\sigma_b \neq \overline{\sigma_b}$.

10 An upper bound for B involving A

Let us use the strategy of §5.2 with the three embeddings σ_b , $\overline{\sigma_b}$ and τ_b .

Lemma 24. *There exists an effectively computable positive constant κ_{47} such that*

$$B \leq \kappa_{47}A.$$

Proof. We use the relation (13) with $\varphi_1 = \sigma_b$, $\varphi_2 = \overline{\sigma_b}$, $\varphi_3 = \tau_b$, in the form $\overline{\sigma_b}(\beta)(\sigma_b(\alpha\varepsilon) - \tau_b(\alpha\varepsilon)) - \sigma_b(\beta)(\overline{\sigma_b}(\alpha\varepsilon) - \tau_b(\alpha\varepsilon)) + \tau_b(\beta)(\overline{\sigma_b}(\alpha\varepsilon) - \sigma_b(\alpha\varepsilon)) = 0$, and we divide by $\sigma_b(\beta)(\overline{\sigma_b}(\alpha\varepsilon) - \tau_b(\alpha\varepsilon))$ (which is not 0):

$$\frac{\overline{\sigma_b}(\beta)}{\sigma_b(\beta)} \cdot \frac{\sigma_b(\alpha\varepsilon) - \tau_b(\alpha\varepsilon)}{\overline{\sigma_b}(\alpha\varepsilon) - \tau_b(\alpha\varepsilon)} - 1 = -\frac{\tau_b(\beta)}{\sigma_b(\beta)} \cdot \frac{\overline{\sigma_b}(\alpha\varepsilon) - \sigma_b(\alpha\varepsilon)}{\overline{\sigma_b}(\alpha\varepsilon) - \tau_b(\alpha\varepsilon)}. \quad (25)$$

The member on the right of (25) is nonzero since $\sigma_b \neq \overline{\sigma}_b$. Let us show that its modulus is bounded from above by

$$e^{\kappa_{48}A} e^{-\kappa_{49}B}.$$

On the one hand, we have

$$|\tau_b(\beta)| \leq e^{-\kappa_{39}B}, \quad |\sigma_b(\beta)| \geq e^{\kappa_{39}B};$$

on the other hand, an upper bound for the absolute logarithmic height of the number

$$\delta = \frac{\overline{\sigma}_b(\alpha\varepsilon) - \sigma_b(\alpha\varepsilon)}{\overline{\sigma}_b(\alpha\varepsilon) - \tau_b(\alpha\varepsilon)}$$

is given by $\kappa_{50}A$, hence $|\delta| \leq e^{\kappa_{48}A}$.

Let us write the term

$$\frac{\overline{\sigma}_b(\beta)}{\sigma_b(\beta)} \cdot \frac{\sigma_b(\alpha\varepsilon) - \tau_b(\alpha\varepsilon)}{\overline{\sigma}_b(\alpha\varepsilon) - \tau_b(\alpha\varepsilon)}$$

which appears on the left side of (25) in the form $\gamma_1^{c_1} \cdots \gamma_s^{c_s}$ with $s = r + 1$, and

$$\gamma_j = \frac{\overline{\sigma}_b(\epsilon_j)}{\sigma_b(\epsilon_j)}, \quad c_j = b_j \quad (j = 1, \dots, r),$$

$$\gamma_s = \frac{\sigma_b(\alpha\varepsilon) - \tau_b(\alpha\varepsilon)}{\overline{\sigma}_b(\alpha\varepsilon) - \tau_b(\alpha\varepsilon)} \cdot \frac{\overline{\sigma}_b(\varrho)}{\sigma_b(\varrho)}, \quad c_s = 1.$$

We have $h(\gamma_s) \leq \kappa_{51}A$.

With

$$H_1 = \cdots = H_r = \kappa_{52}, \quad H_s = \kappa_{52}A,$$

we have, using Lemma 21,

$$\max_{1 \leq j \leq s} \frac{H_j}{H_s} |c_j| \leq \frac{\kappa_{53}B}{H_s}.$$

Moreover, with

$$C_1 = 2 + \frac{B}{A},$$

we obtain from Corollary 10 that a lower bound of the modulus of the left member of (25) is given by $\exp\{-\kappa_{54}H_s \log C_1\}$. The above upper bound and the last lower bound lead to

$$\exp\{-\kappa_{54}H_s \log C_1\} \leq \exp\{-\kappa_{49}B + \kappa_{48}A\}.$$

Consequently,

$$\kappa_{49}B \leq \kappa_{48}A + \kappa_{54}H_s \log C_1.$$

Therefore,

$$2 + \frac{B}{A} = C_1 \leq \kappa_{55} \log C_1,$$

which allows to conclude $C_1 \leq \kappa_{56}$, namely $B/A \leq \kappa_{47}$. \square

11 Ultimate diophantine argument

We will use the strategy of §5.2 with the three embeddings $\sigma_a, \overline{\sigma}_a, \tau_b$.

Lemma 26. *We have*

$$\frac{A+B}{\log m} \leq \kappa_{57} \log \frac{A+B}{1+\log m}.$$

Proof. We use the relation (13) with $\varphi_1 = \sigma_a$, $\varphi_2 = \overline{\sigma}_a$, $\varphi_3 = \tau_b$ under the form

$$\frac{\sigma_a(\alpha\varepsilon)\overline{\sigma}_a(\beta)}{\overline{\sigma}_a(\alpha\varepsilon)\sigma_a(\beta)} - 1 = \frac{\tau_b(\beta)}{\sigma_a(\beta)} \left(\frac{\sigma_a(\alpha\varepsilon)}{\overline{\sigma}_a(\alpha\varepsilon)} - 1 \right) + \frac{\tau_b(\alpha\varepsilon)}{\overline{\sigma}_a(\alpha\varepsilon)} \left(\frac{\overline{\sigma}_a(\beta)}{\sigma_a(\beta)} - 1 \right). \quad (27)$$

The left member of (27) is not 0 since $\sigma_a \neq \overline{\sigma}_a$. The inequalities (19) and Lemma 20 indicate that we have

$$|\tau_b(\beta)| \leq e^{-\kappa_{39}B}, \quad |\tau_b(\alpha\varepsilon)| \leq 2, \quad |\sigma_a(\beta)| = |\overline{\sigma}_a(\beta)| \geq e^{\kappa_{40}A}.$$

Since

$$\max \left\{ \left| \frac{\sigma_a(\alpha\varepsilon)}{\overline{\sigma}_a(\alpha\varepsilon)} - 1 \right|, \left| \frac{\overline{\sigma}_a(\beta)}{\sigma_a(\beta)} - 1 \right| \right\} \leq 2,$$

we deduce that each of the two terms

$$\frac{\tau_b(\beta)}{\sigma_a(\beta)} \left(\frac{\sigma_a(\alpha\varepsilon)}{\overline{\sigma}_a(\alpha\varepsilon)} - 1 \right) \quad \text{and} \quad \frac{\tau_b(\alpha\varepsilon)}{\overline{\sigma}_a(\alpha\varepsilon)} \left(\frac{\overline{\sigma}_a(\beta)}{\sigma_a(\beta)} - 1 \right)$$

of the right member of (27) has a modulus bounded from above by $e^{-\kappa_{58}A}$.

Let us write

$$\frac{\sigma_a(\alpha\varepsilon)\overline{\sigma}_a(\beta)}{\overline{\sigma}_a(\alpha\varepsilon)\sigma_a(\beta)} = \gamma_1^{c_1} \cdots \gamma_s^{c_s}$$

with $s = r + 1$ and

$$\gamma_i = \frac{\sigma_a(\epsilon_i)}{\overline{\sigma}_a(\epsilon_i)}, \quad c_i = a_i - b_i \quad (i = 1, \dots, r), \quad \gamma_s = \frac{\sigma_a(\alpha\zeta)\overline{\sigma}_a(\varrho)}{\overline{\sigma}_a(\alpha\zeta)\sigma_a(\varrho)}, \quad c_s = 1.$$

The absolute logarithmic height of γ_s is bounded from above by

$$h(\gamma_s) \leq \kappa_{59} \log m.$$

Let us also write

$$H_1 = \cdots = H_r = \kappa_{60}, \quad H_s = \kappa_{60}(1 + \log m), \quad C_1 = 2 + \frac{A+B}{1+\log m}.$$

Corollary 10 implies that the modulus of the left member of (27) has a lower bound given by $\exp\{-\kappa_{61}H_s \log C_1\}$. The above upper bound and the last lower bound lead to

$$\exp\{-\kappa_{61}H_s \log C_1\} \leq 2 \exp\{-\kappa_{58}A\}.$$

Consequently,

$$\frac{A+B}{\log m} \leq \kappa_{62} \log C_1.$$

This completes the proof of Lemma 26. \square

12 End of the proof of Theorem 2

Thanks to Lemma 21 we have

$$A \leq \kappa_{41} B$$

while from Lemma 24 we deduce

$$B \leq \kappa_{47} A,$$

and from Lemma 26 we have

$$\frac{A+B}{\log m} \leq \kappa_{57} \log \frac{A+B}{1+\log m}.$$

We conclude $\max\{A, B\} \leq \kappa_{63} \log m$, and this is what we wanted to show in order to apply Lemma 14.

References

- [1] C. LEVESQUE AND M. WALDSCHMIDT, *Familles d'équations de Thue–Mahler n'ayant que des solutions triviales*, Acta Arith., **155** (2012), 117–138.
- [2] C. LEVESQUE AND M. WALDSCHMIDT, *Families of cubic Thue equations with effective bounds for the solutions*, Number Theory Meeting in Australia 2012, in memory of Alf van der Poorten, Springer Verlag (to appear).
- [3] T.N. SHOREY AND R. TIJDEMAN, *Exponential Diophantine equations*, vol. 87 of Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 1986.
- [4] H.-J. STENDER, *Lösbare Gleichungen $ax^n - by^n = c$ und Grundeinheiten für einige algebraische Zahlkörper vom Grade $n = 3, 4, 6$* , J. reine angew. Math. **290**, (1977), 24–62.
- [5] M. WALDSCHMIDT, *Diophantine approximation on linear algebraic groups*, volume **326** of *Grundlehren der Mathematischen Wissenschaften* (Springer-Verlag, Berlin, 2000).